



# The Extended Thermodynamic Phase Structure of Taub–NUT and Taub–Bolt

Clifford V. Johnson

*Department of Physics and Astronomy  
University of Southern California  
Los Angeles, CA 90089-0484, U.S.A.*

johnson1, [at] usc.edu

## Abstract

We study aspects of the extended gravitational thermodynamics for the Taub–NUT and Taub–Bolt geometries in four dimensional locally anti-de Sitter spacetime, where the cosmological constant is treated as a dynamical pressure. Attention is paid to the phase structure in the  $(p, T)$  plane, which has a line of first order phase transitions extending from the origin. We argue for a dynamical interpretation of the unstable physics in the negative specific heat region. A deformation corresponding to a dyonic Taub–NUT/Bolt system is also explored, and the effect of the deformation on the phase diagram is characterised.

# 1 Motivation

Semi-classical quantum gravity has its most beautiful realisation in the context of black hole physics, where it breathes life into the equations describing their thermodynamics [1–4]. The standard central quantities are the black hole’s mass  $M$ , surface gravity  $\kappa$ , and area  $A$ , which are related to the thermodynamic energy  $U$ , temperature  $T$ , and entropy  $S$ , respectively. An odd (but perfectly consistent) feature of this framework is the absence of a variable representing pressure and its conjugate, volume, two central players in traditional discussions of thermodynamics when applied to everyday substances. Recent work<sup>1</sup> has shown that the effective cosmological constant,  $\Lambda$ , of the spacetime in question can be treated as a pressure *via*  $p = -\Lambda/8\pi G$ , ( $G$  is Newton’s constant, and we will choose units such that  $c = 1$ ,  $\hbar = 1$  and  $k_B = 1$ ), and study of the resulting “extended thermodynamics” results in the natural suggestion [9] that black hole mass  $M$  determines not the internal energy  $U$  but the *enthalpy*:  $M/G = H \equiv U + pV$ , which includes a contribution from the energy of formation of the system. The thermodynamic volume  $V$  can then be deduced in terms of the variables of the black hole spacetime in question.

One interesting consequence of this is the observation that  $V$  is not necessarily a geometrical volume defined in the spacetime at all, a fact revealed by *e.g.*, studying rotating black holes [11]. Motivated by the AdS/CFT correspondence [19–22], and by the fact that consistent traditional thermodynamics can be defined for non-black-hole spacetimes (see *e.g.* refs. [23, 24]), the suggestion was made in ref. [25] to enlarge the framework of extended thermodynamics to include *any* spacetime, again equating the mass to the enthalpy. The result was a larger class of examples of thermodynamic volumes which are non-geometrical (in the above sense), now associated to the Taub–NUT and Taub–Bolt spacetimes<sup>2</sup>. The extreme example of Taub–NUT is especially notable for having a thermodynamic volume while there is no geometrical candidate at all. We summarise some of the results we will need in sections 2 and 3.

In a sense, the extended thermodynamics brings the physics of black holes closer to the physics of everyday (non-gravitational) substances in a manner complementary to way the AdS/CFT correspondence does [21, 22]. For example, using the standard treatment of the thermodynamics, it was discovered [27, 28] that the fixed charge ensemble of Reissner–Nordstrom black holes in AdS has a rich thermodynamic phase structure reminiscent of the van der Waals description of a liquid/gas system, including a line of first order phase transitions ending in a second order critical point. Through AdS/CFT this maps holographically to a strongly coupled fluid in an unusual universal-

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<sup>1</sup>For a selection of references, see refs. [5–13], including the reviews in refs. [14, 15]. See also the early work in refs. [16–18].

<sup>2</sup>See also ref. [26] for an example mixing the features of refs. [11, 25] by combining nut charge and rotation.

ity class. This is of course an exciting and important way of defining non-traditional universality classes of fluids and other substances, given its potential use for interpreting and modelling new experimental discoveries. This is an idea that is well over a decade old [29, 30]. On the other hand it is intriguing to learn that black hole physics can be mapped (at least in part) to more traditional classes of substance, which is precisely what the extended thermodynamics has made possible. For example, that same charged black hole system, once pressure and volume are allowed to be dynamical, turns out to have the same critical properties as the van der Waals system<sup>3</sup> including the standard mean field exponents. Several other familiar thermodynamic phase diagram phenomena (many in traditional universality classes) have now been mapped to black hole physics in the extended thermodynamics. Work on this continues by several groups<sup>4</sup>.

The meaning of the extended thermodynamics and what it implies for systems described holographically by AdS/CFT remains to be spelled out. It is likely that the two areas can enrich each other once the right interpretation is found. As suggested in ref. [33], it seems clear that the natural AdS/CFT context where the extended thermodynamics might play a role would concern situations where there is flow between field theories, perhaps involving operators that can dynamically change the effective degrees of freedom. Perhaps the extended thermodynamics is a good language for discussing dynamics on the space of field theories itself, and in the AdS/CFT language this boils down to dynamics on the space of holographic RG flows, a quite natural object in gauged supergravity, and perhaps string theory. Intriguingly, natural thermodynamic principles, such as the Second Law of thermodynamics (realised in terms of thermodynamic cycles representing heat engines and refrigerators) would translate into statements about the kinds of processes one could do in the space of field theories<sup>5</sup>.

One of the other points made in ref. [33] is that the enlarged dynamical setting of the extended thermodynamics helps place sometimes puzzling thermodynamic properties of spacetimes into a light where they can be seen to make physical sense. For example,  $\text{AdS}_4$  with an  $S^3$  slicing (where Euclidean time is an  $S^1$  fibred over an  $S^2$ ) has negative entropy. For fixed cosmological constant this is simply strange. On the other hand,  $\text{AdS}_4$  is a perfectly good spacetime that has just been sliced unusually. Rather than discard it, it would seem prudent to simply find a context in which this slicing makes sense. Putting it into the extended thermodynamics context and using the enthalpy prescription reveals that it has a negative thermodynamic volume [33], which might also

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<sup>3</sup>See ref. [31] and also refs. [5, 13].

<sup>4</sup>For recent reviews, see refs. [15, 32].

<sup>5</sup>This suggestion of ref. [33] was based on the observation that the cosmological constant (and hence  $p$ ) is connected to the number of degrees of freedom in the dual field theory by virtue of being related to the rank,  $N$ , of the dual gauge theory. Note that although they did not pursue the connection, refs. [9, 34] also mention the link between  $p$  and  $N$ , and wondered as to its significance. We thank B. Dolan and D. Kastor for pointing this out.

seem odd. However, in a context where there are dynamically changing pressures and volumes, and accompanying heat flow, both work done and heat flow into the volume naturally come with both signs, and so for positive  $T$  and  $p$  can result in negative  $S$  and  $V$ , with the correct interpretation.

A negative specific heat should have a natural dynamical narrative as well. It is familiar, for example in the case of Schwarzschild black holes in flat space. In extended thermodynamics it becomes the specific heat at constant pressure<sup>6</sup>,  $C_p$ . Negative specific heat suggests a thermal instability, and so we are strictly outside equilibrium thermodynamics now, but can at least start to follow the physics. The system runs to higher temperature (like a flat space black hole radiating away its energy), lowering its entropy. This is a rather runaway process for a black hole in flat space, and so a discussion of it is somewhat incomplete since it runs off to regimes that lie well outside the semi-classical context in which we began. Adding a cosmological constant avoids the problem altogether, rather than solving it: Analogues of the Schwarzschild black holes exist in AdS as the solutions representing the “small” branch of black holes, but they never appear in the phase diagram, (either in the standard or extended thermodynamics). At any pressure (and hence  $\Lambda$ ) the low temperature phase is AdS (with the traditional  $S^1 \times S^2$  slicing), and then there is the Hawking–Page transition [35] temperature above which the “large” branch black holes are thermodynamically favoured. So the negative  $C_p$  of the black holes never needs to be interpreted.

Nevertheless, it seems natural to try to give negative  $C_p$  an interpretation in other systems where it might not be avoided, with the help of the extended thermodynamics. We have a context in which to do so: Start again with AdS<sub>4</sub> with the  $S^3$  slicing. It lies on an extended phase diagram that we will explore more in this paper. We are free to adjust the temperature of this slicing by viewing AdS<sub>4</sub> as part of the Taub–NUT–AdS class of spacetimes [36], characterised by nut charge  $n$ , with temperature  $T = 1/8\pi n$ . The case  $n = \ell/2$  (where  $\ell$  is the length scale set by the cosmological constant  $\Lambda = -3/\ell^2$ ) is AdS<sub>4</sub>. The Taub–NUT–AdS spaces have [37, 38] a characteristic temperature,  $T^o$ , above which  $C_p < 0$ .<sup>7</sup> At another temperature  $T^*$  above  $T^o$ , there is a transition to the (“large” branch) Taub–Bolt spacetime, which is stable [36]. Notice that this leaves a window  $T^o < T < T^*$  where the geometry with lower action has  $C_p < 0$ . This demands to be treated as interpretable sensible physics, and we do so. It is in a sense a “safer” system than the flat space Schwarzschild black hole case since the physics does not run away to regimes outside the semi-classical domain; the Taub–bolt spacetime takes over, restoring the equilibrium thermodynamics.

It is in this spirit we present and discuss the  $(p, T)$  phase diagram of the Taub–NUT/Taub–

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<sup>6</sup>The counterpart specific heat,  $C_V$ , vanishes for static black holes [10].

<sup>7</sup>They also have lower temperature below which  $S < 0$ , but we will not concern ourselves much with this region of parameter space.

Bolt system, in section 4. It is also natural to consider a deformation of our system by electric and magnetic fields that make the NUT/Bolt solutions into dyons. The electric and magnetic charges of the solutions (discussed in section 5) will depend upon one parameter,  $v$ , and it will be natural to study both real and imaginary values of this parameter, unpacking the thermodynamic phase structure. We discuss the resulting phase diagrams in section 6.

## 2 Taub–NUT–AdS and Taub–Bolt–AdS

We work with Einstein–Hilbert action:

$$I = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) , \quad (1)$$

with negative cosmological constant  $\Lambda$  which sets a length scale  $\ell$ :  $\Lambda = -3/\ell^2$ . The Taub–NUT [39,40] and Taub–Bolt [41] spacetimes, have a metric of the following form for negative cosmological constant [42,43]:

$$ds^2 = F(r)(d\tau + 2n \cos \theta d\phi)^2 + \frac{dr^2}{F(r)} + (r^2 - n^2)(d\theta^2 + \sin^2 \theta d\phi^2) , \quad (2)$$

where

$$F(r) \equiv \frac{(r^2 + n^2) - 2mr + \ell^{-2}(r^4 - 6n^2r^2 - 3n^4)}{r^2 - n^2} . \quad (3)$$

Time  $\tau$  here is Euclidean with period  $\beta = 8\pi n$  (in order to ensure the physical absence of Misner strings [44]). The spaces are locally asymptotically  $\text{AdS}_4$ , generically; The topological  $S^3$  formed by the  $\tau$  circle fibred over the  $S^2$  of  $\theta$  and  $\phi$  is squashed for general  $n$ . The case  $n = \ell/2$  in fact yields  $\text{AdS}_4$ , with radial slices that are round  $S^3$ s [36], as discussed in the previous section.

The circle parameterized by  $\tau$  is Hopf fibred over the  $S^2$ . It degenerates at some radius  $r_+$  when  $F$  vanishes, in which case:

$$m = \frac{r_+^2 + n^2}{2r_+} + \frac{1}{2\ell^2} \left( r_+^3 - 6n^2r_+ - 3\frac{n^4}{r_+} \right) . \quad (4)$$

To ensure that there is no conical singularity, we have  $F'(r = r_+) = 1/2n$ . The parameters  $m$  and  $n$ , and the range of  $r$  have certain allowed ranges that depend upon whether we are Taub–NUT or Taub–Bolt, and the conditions have been worked out in the literature [36]:

- For the Taub–NUT case  $r_+ = r_n = n$ . The spacetime there is locally  $\mathbb{R}^4$ , a “nut” [45]. The mass parameter  $m = m_n$ , and the equation (4) simplifies to:

$$m_n = n - \frac{4n^3}{\ell^2} , \quad (5)$$

and we have that  $n \leq r \leq +\infty$ .

• For the Taub–Bolt case  $r_+ = r_b > n$ , and it satisfies a quadratic equation:

$$6nr_b^2 - \ell^2 r_b - 6n^3 + 2n\ell^2 = 0, \quad (6)$$

with an upper ( $r_{b+}$ ) and lower ( $r_{b-}$ ) branch solution:

$$r_{b\pm} = \frac{\ell^2}{12n} \left( 1 \pm \sqrt{1 - 48\frac{n^2}{\ell^2} + 144\frac{n^4}{\ell^4}} \right), \quad (7)$$

with  $n$  restricted to be no greater than  $n_{\max}$  (where the branches join) in order to have real  $r_b$  greater than  $n$ :

$$n_{\max} = \left( \frac{1}{6} - \frac{\sqrt{3}}{12} \right)^{\frac{1}{2}} \ell. \quad (8)$$

The form of the function  $r_b(n)$  is plotted in figure 1.

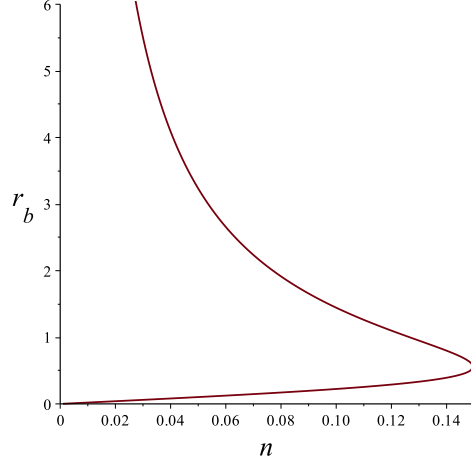


Figure 1: The available bolt radii, the union of an upper and a lower branch that connect at  $n_{\max} \simeq 0.149\ell$ . Here we chose  $\ell = 1$ .

The spacetime in the neighbourhood of  $r = r_b$  is locally  $\mathbb{R}^2 \times S^2$ , a “bolt” [45]. In that case the mass parameter  $m = m_b$ , where we use the mass formula (4) with  $r_+ = r_b$ , and we have that  $r_b \leq r \leq +\infty$ . The upper and lower branches are analogous to the large and small black hole branches possessed by AdS–Schwarzschild black holes [35]. (The computations are structurally different, but with similar results. For black holes the horizon radius  $r_+$  is given and then consistency yields a  $\tau$  period  $\beta(r_+)$ . Here, the period  $8\pi n$  is given, and then consistency<sup>8</sup> produces a function  $r_b(n)$ .) The two branches’ characteristics become sharply distinct for small  $n$  (equivalently large  $\ell$ ), where

$$r_{b-} \simeq 2n + \frac{18n^3}{\ell^2} + O(n^5), \quad r_{b+} \simeq \frac{\ell^2}{6n} - 2n - \frac{18n^3}{\ell^2} + O(n^5), \quad (9)$$

showing that the lower branch, constituted of bolts whose radii are “small” compared to the length scale  $\ell$  set by the cosmological constant, connect smoothly to the asymptotically locally flat ( $\Lambda = 0$ ) case.

<sup>8</sup>With the temperature  $T = 1/8\pi n$  for AdS–Taub–NUT/Bolt, one might wonder about the case  $n = 0$  which reduces the metric (2) back to AdS–Schwarzschild. The latter has a temperature that depends explicitly on the mass. There is no problem with the  $n \rightarrow 0$  limit, where for AdS–Taub–NUT/Bolt,  $T \rightarrow \infty$ . One should also follow the mass formulae (4) and (5), which show that  $m$  vanishes or (for the large  $r_{b+}$  branches) diverges as  $n \rightarrow 0$ . This perfectly matches the  $T \rightarrow \infty$  behaviour of the small and large black hole AdS–Schwarzschild solutions that the  $n = 0$  metric yields for those mass values.

### 3 Thermodynamic Variables

The temperature of these spacetimes is set by the inverse of  $\tau$ 's period,  $T = 1/8\pi n$ . The action of each was computed in refs. [37, 38] and may be written in the very simple form given in ref. [38]:

$$I = \frac{4\pi n}{G\ell^2} (\ell^2 m + 3n^2 r_+ - r_+^3) . \quad (10)$$

For Taub–NUT we put  $m = m_n$  and  $r_+ = n$  while for the Taub–Bolt we use  $m = m_b$  and  $r_+ = r_b$ . The difference between the actions of the two spacetimes (which was actually computed before the expression above was known, by using a subtraction method [36, 46]), can be written as:

$$I_b - I_n = -\frac{2\pi n}{G\ell^2} \left( \frac{(r_b - n)^2 (r_b^2 - 6nr_b - n^2)}{r_b - 2n} \right) . \quad (11)$$

The point here is that it is natural, at a given  $n$ , to compare the action of Taub–NUT to that of Taub–Bolt since they have the same asymptotics at large  $r$ . The previous work on the phase structure derived by examining this action difference was done at fixed cosmological constant. In the extended thermodynamics  $\Lambda = -3/\ell^2$  defines a pressure  $p = 3/(8\pi G\ell^2)$  and so we have instead a line of phase transition points. We will discuss this in a later section.

The entropy can be derived from the action using  $S = (\beta\partial_\beta - 1)I$ , and after substituting to make the pressure more explicit we have the form [25]:

$$S = \frac{4\pi n}{G} \left[ \frac{r_+^2 + n^2}{2r_+} + \frac{4\pi G p}{3} \left( 3r_+^3 - 12n^2 r_+ - \frac{3n^4}{r_+} \right) \right] .$$

Note that there is in principle  $\ell$  dependence and hence pressure dependence in the  $r_+$  function as well. It is only in the case of Taub–NUT, where  $r_+ = r_n = n$ , that things simplify to give a very simple  $p$  dependence, or in the case of small bolts, where  $r_+ \rightarrow 2n$  and we return in the limit to zero cosmological constant. We will discuss these cases below.

Finally, in the extended thermodynamics it was proposed in ref. [25] that the mass for the Taub–NUT and Taub–Bolt spacetimes define the enthalpy of the system, and so we write:  $H = m/G$ :

$$H(S, p) = \frac{1}{G} \frac{r_+^2 + n^2}{2r_+} + \frac{4\pi}{3} p \left( r_+^3 - 6n^2 r_+ - 3\frac{n^4}{r_+} \right) , \quad (12)$$

and this results in an interesting expression for the conjugate to pressure, the thermodynamic volume [25]:

$$V = \frac{4\pi}{3} (r_+^3 - 3n^2 r_+) . \quad (13)$$

Note that this thermodynamic volume is (in general) not to be associated directly with any geometric volume in the spacetime, as discussed in detail in ref. [25]. It can also be negative, which

was also given an interpretation in ref. [25]: the environment did positive work in forming the spacetime, as opposed to having work done on it, as is more familiar with an enthalpy.

The volume for Taub–NUT is  $\ell$ -independent and so is also true in asymptotically locally flat space (the limit of  $p = 0 = \Lambda$ ). Interestingly, the result for the Taub–Bolt spacetime in asymptotically flat space (using at large  $\ell$  the small branch family:  $r_{b-} \rightarrow 2n$  in the limit) is equal in magnitude to that of Taub–NUT, with opposite sign:

$$V_b|_{p=0} = \frac{8\pi n^3}{3} = -V_n . \quad (14)$$

The NUT and Bolt spacetimes seem rather complementary in their roles. In this limit, it is worth noting the other (finite) quantities as well [47]:

$$\begin{aligned} H_n|_{p=0} &\equiv \frac{m_n}{G}|_{p=0} = \frac{n}{G} ; & I_n|_{p=0} &= \frac{4\pi n^2}{G} ; & S_n|_{p=0} &= \frac{4\pi n^2}{G} . \\ H_b|_{p=0} &\equiv \frac{m_b}{G}|_{p=0} = \frac{5}{4} \frac{n}{G} ; & I_b|_{p=0} &= \frac{5\pi n^2}{G} ; & S_b|_{p=0} &= \frac{5\pi n^2}{G} . \end{aligned} \quad (15)$$

## 4 Phase Structure

The phase structure of the Taub–NUT/Taub–Bolt system, as studied in ref. [36, 46], was only the structure at a particular value of the cosmological constant, and hence at fixed pressure. The action difference between each spacetime was studied as a function of temperature. At a certain temperature, the Taub–Bolt spacetime had lower action, and so there was a first order transition where the difference  $I_b - I_n$  in equation (11) becomes negative. Since there are two branches of bolts, the curve of  $I_b - I_n$  as a function of  $n$  (which sets the inverse temperature) is made of two parts that meet (at  $n_{\max}$ ) to form a cusp shape. The transition is for the large bolts (the  $r_{b+}$  branch).

In the extended thermodynamics where pressure is a variable, we can explore the phase transitions found in the earlier work further, seeing that they form a line of points in the  $(p, T)$  plane. Each value of the pressure  $p = 3/(8\pi\ell^2 G)$  gives a cusp curve where the cusp starts at  $n_{\max}$  and proceeds on two branches to

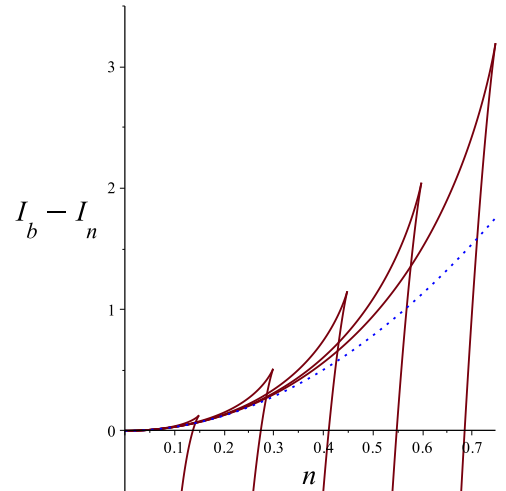


Figure 2: The action (or Gibbs free energy) difference for pressures  $p = 3/(8\pi\ell^2 G)$ , for  $\ell = 1, 2, 3, 4, 5$ , with  $G = 1$ . The curves move more to the right as the pressure decreases. The blue dotted line is the case of zero pressure, which the solid curves approach in the limit.



lower  $n$ . In the limit of infinite  $\ell$  (zero pressure), it is only the part of the cusp that corresponds to the small bolts (the  $r_{b-}$  branch) that has finite action, and for them  $I_b - I_n = \pi n^2/G$ . See figure 2 and equation (15). (Parenthetically, notice that the curves can all be superimposed by a scaling. This actually follows from equation (11) for the action difference. If  $n \rightarrow \alpha n$  and  $\ell \rightarrow \alpha \ell$ , then  $(I_b - I_n) \rightarrow \alpha^2(I_b - I_n)$ . This all results from the fact that an accompanying scaling of the coordinates ( $\tau \rightarrow \alpha \tau$  and  $r \rightarrow \alpha r$ ) rescales<sup>9</sup> the overall metric of equation (2):  $ds^2 \rightarrow \alpha^2 ds^2$ .)

The action difference (11) changes sign at the positive zero of the quadratic in the numerator, which is at  $r_b = (3 + \sqrt{10})n$ . Putting this into the quadratic equation (6) satisfied by  $r_b$  gives an equation for the value of  $n$  at which the transition happens, which we denote as  $n^*$ . After some algebra,

$$n^* = \frac{1}{6}(\sqrt{5} - \sqrt{2})\ell, \quad (16)$$

and further algebra can be used to turn this into an expression for the transition temperature  $T^*$ , as a function of pressure:

$$T^*(p) = \frac{\sqrt{3G}}{\sqrt{2\pi}(\sqrt{5} - \sqrt{2})} p^{\frac{1}{2}}. \quad (17)$$

So far this is an analogue of a solid–liquid coexistence curve<sup>10</sup> in the phase diagram, although further study in this section will show that only one phase can exist near the line, due to a thermal instability. Then section 6 will study a deformation that will allow for regimes where it is genuinely a coexistence line.

The  $p = 0$  point on the curve is deceptive. In fact, strictly at zero pressure, there is no bolt phase at all, as the large bolts have gone to infinite size, and their action is infinite. But we see now in this larger context that this is a very special point that neighbours on a richer story. The transition temperature is lower for lower pressure, and so as soon as there is some non-zero cosmological constant (pressure) there will be a nearby transition to a bolt solution.

This is analogous to the Hawking–Page black hole transition, where at finite cosmological constant there is a transition from pure AdS to an AdS–Schwarzschild black hole, but that transition disappears at zero cosmological constant: Flat space Schwarzschild black holes (the small branch survivors of the two branches available at finite  $\Lambda$ ) have higher action than flat space at any temperature. In fact, the varying pressure coexistence curve can be computed in that case too (see ref. [15] for a recent review), with the same functional dependence, but with coefficient  $\sqrt{8/3\pi}$  instead of the one displayed in equation (17) for our case.

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<sup>9</sup>We thank an anonymous referee for highlighting this scaling.

<sup>10</sup>This  $T \sim p^{1/2}$  form of the curve, and other phase boundaries in the  $(p, T)$  plane that we will uncover, follows from the scaling properties discussed at the end of the previous paragraph, since  $T \rightarrow T/\alpha$  and  $p \rightarrow p/\alpha^2$ .

The diagram has two additional curves plotted on it as well. The shaded region between them has positive values for both the entropy and the specific heat at constant pressure,  $C_p = T \partial S / \partial T|_p$ . We'll call this the “positive wedge” in the  $(p, T)$  phase diagram, for the purposes of discussion. Along a fixed pressure slice, for low enough temperatures, the entropy is negative, while at high enough temperatures  $C_p < 0$ . Between these two temperatures, both are positive. The temperatures can be read off from equation (12) specialised to the Taub–NUT entropy (by putting  $r_+ = n$ ) to give:

$$S_n = \frac{4\pi n^2}{G} (1 - 16\pi G p n^2) , \quad (18)$$

from which the specific heat can be computed as<sup>11</sup>

$$C_p = \frac{8\pi n^2}{G} (32\pi G p n^2 - 1) , \quad C_V = 0 . \quad (19)$$

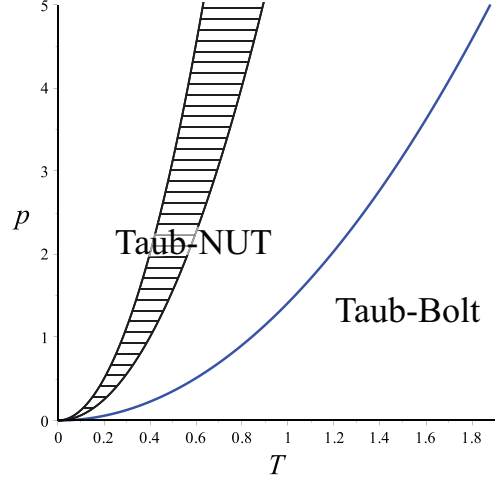


Figure 3: The rightmost solid curve represents the line of transition between the Taub–NUT and Taub–Bolt phases. The shaded region between the two curves on the left, referred to as the “positive wedge” in the text, is where entropy  $S > 0$  and specific heat  $C_p > 0$ . For points in the region between the positive wedge and the transition, the system is dynamically driven by fluctuations toward the transition line.

From the above expressions we can compute the profile of the curves that border the positive wedge. At a given value of the pressure  $p$ , we have for the lower temperature:

$$T^\# \equiv \frac{1}{8\pi n^\#} = \left( \frac{G}{4\pi} \right)^{\frac{1}{2}} p^{\frac{1}{2}} , \quad (20)$$

and for the upper temperature:

$$T^o \equiv \frac{1}{8\pi n^o} = \left( \frac{G}{2\pi} \right)^{\frac{1}{2}} p^{\frac{1}{2}} , \quad (21)$$

and so we have  $T^\# \leq T \leq T^o$  for the slice of the positive wedge located at pressure  $p$ . After the transition temperature  $T^*$ , the entropy of the favoured bolt solutions is both positive and has  $C_p > 0$ . Figure 4 shows the entropy as a function of  $T$  for a fixed pressure slice, and the three temperatures  $T^\#, T^o, T^*$ .

The negative entropy of Taub–NUT for the region  $T < T^\#$  and the negative specific heat for  $T$  in the region between  $T^o$  and the transition at  $T^*$  are identified as puzzles in refs. [37,38]. There is a very natural narrative in the context here, *i.e.*, extended thermodynamics. For a start, having

<sup>11</sup>The latter result follows since the Taub–NUT thermodynamic volume  $|V_n| = 8\pi n^3/3$  depends strictly on the temperature only through  $T = 1/8\pi n$  and hence fixing the volume fixes  $S$ . This is analogous to the vanishing of  $C_V$  for static black holes, although their origins are quite different. There, the entropy and volume are not independent quantities, both depending upon a single parameter, the radius  $r_h$  of the black hole.

a negative entropy for the spacetime can quite naturally arise as a result of some net heat outflow during the same formation process that accompanies the interpretation of the mass as enthalpy of formation, as discussed in ref. [25]. We won't have any more to say about that here, except to note that we can, if we choose to, avoid the negative entropy region altogether by appropriate paths in the  $(p, T)$  plane. For example, one can start at  $p = 0$  and turn on finite temperature. The entropy is positive and given in equations (15). The same set of equations shows that the action of Taub–NUT is lower than Taub–Bolt and so is thermodynamically favoured. Then one can directly enter the positive wedge by turning on small enough pressure  $p$ . We can move around inside the positive wedge and do standard equilibrium thermodynamics. Notice that the special case when Taub–NUT is actually pure  $\text{AdS}_4$  with an  $S^3$  slicing,  $n = \ell/2$ , *i.e.*  $T = \sqrt{(G/6\pi)}p^{1/2}$ , lies outside this wedge.

Now we wish to consider how to traverse the region in figure (3) between the positive wedge in the Taub–NUT region and the transition at  $T^*$  to Taub–Bolt. Are there any obstructions to doing so? The fact that  $C_p$  goes negative at the edge of the wedge signals what presumably happens. The system develops an instability to fluctuations above the edge temperature  $T^o = \sqrt{(G/2\pi)}p^{1/2}$ , and it is dynamically predisposed to drive itself to higher temperature, where the specific heat becomes even more negative, improving the susceptibility to instability. This happens until the system arrives at  $T^*$  and transitions to Taub–Bolt, where it is again stable.

We can go further and track the changes in the system through this process in the extended thermodynamics. It is easy to do at fixed pressure, *i.e.*, moving along a state curve of the form given in figure 4. This isobaric process has a volume change, and so (in the spirit of ref. [33]) we should take this seriously as a physical process during which work is done either on or by the system, with an accompanying heat flow. This changes the internal energy of the system, which was computed in ref. [25] to be:

$$U_n = \frac{n}{G} (1 - 8\pi G p n^2) . \quad (22)$$

All of this is of course accounted for using the First Law of thermodynamics, naturally written in

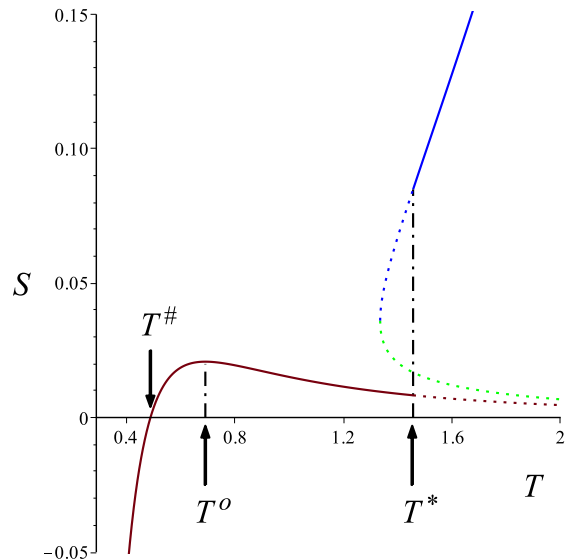


Figure 4: The entropy curves as a function of temperature, at fixed pressure. Here  $p = 3$ . As  $T$  increases, there is a transition from Taub–NUT to Taub–Bolt at  $T^*$ . Dotted lines show solutions with higher action.  $T^\#$  and  $T^o$  are discussed in the text.

terms of the enthalpy  $H_n = U_n + pV_n$  as  $dH_n = TdS_n + V_n dp$ . Our isobar gives us:

$$\begin{aligned}
\Delta H_n &= \int_{n^o}^{n^*} \frac{1}{8\pi n} \frac{\partial S_n}{\partial n} dn = \frac{1}{G}(n^* - n^o) - \frac{32\pi p}{3}[(n^*)^3 - (n^o)^3] \\
&= \frac{1}{G}(n^* - n^o) - 8\pi p[(n^*)^3 - (n^o)^3] - p\frac{8\pi}{3}[(n^*)^3 - (n^o)^3] \\
&= \Delta U_n + p\Delta V_n .
\end{aligned} \tag{23}$$

In summary then, it is natural to think of the pressure as a parameter that allows us to explore a larger phase space of parameters that deform the Taub–NUT/Bolt system away from flat space (the origin). A critical line develops separating Taub–NUT and Taub–Bolt phases, and two regions develop in the Taub–NUT area, one has  $S < 0$  and the other has  $C_p < 0$ , separated by the “positive wedge”. We can avoid the  $S < 0$  region by appropriate choice of path in the  $(p, T)$  plane. Fluctuations dynamically drive points in the  $C_p < 0$  region toward the transition line to become Taub–Bolt solutions. Presumably this means that one cannot quasi-statically follow a path from the Taub–Bolt region back toward Taub–NUT solutions<sup>12</sup>. Furthermore this means that the transition line is strictly not a phase coexistence line like those familiar in solid/liquid transitions. The  $C_p < 0$  region means that the nuts and bolts, despite having equal action (Gibbs free energy) there, do not coexist indefinitely. Taub–Bolt dominates after a long enough time.

However, the relative locations of the positive wedge region and the transition line are adjustable using a natural deformation parameter. We explore that in the next two sections, finding that there are some values of the parameter where the transition line is more like a standard coexistence line.

## 5 Dyonic Taub–NUT–AdS and Taub–Bolt–AdS

The Taub–NUT/Bolt spacetime has many interesting properties. One of them is that while it possesses ordinary mass, it also has nut charge, which can in fact be interpreted as a magnetic counterpart to the ordinary “electric” mass. In this sense it is a dyon, for the mass sector, and this helps contribute to Taub–NUT’s role as a self-dual gravitational instanton [24], where the Euclidean section sets the mass  $m$  equal to the nut charge  $n$ .

One might wonder about the fate of Taub–NUT if one embeds it non-trivially into electromagnetism, extending the Einstein–Hilbert action to incorporate Maxwell:

$$I = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda - F^2) , \tag{24}$$

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<sup>12</sup>So if constructing holographic heat engines using these spacetimes as a “working substance”, as done for black holes in ref. [33], the cycles cannot include paths that traverse the  $C_p < 0$  region.

with  $F$  (not to be confused with the metric function in equation (2)) the standard gauge field two-form with components  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . It was shown in ref. [48] that there is a charged geometry, and in fact both an electric and magnetic field are turned on. We have a dyon in the electromagnetic sense as well<sup>13</sup>. A more general solution can be written down that has non-zero cosmological constant, and it has the same form for the metric given in equation (2), but with a more general form for the metric function  $F(r)$  given by [51–53]:

$$F(r) \equiv \frac{(r^2 + n^2 + 4n^2v^2 - q^2) - 2mr + \ell^{-2}(r^4 - 6n^2r^2 - 3n^4)}{r^2 - n^2} , \quad (25)$$

and the gauge sector is:

$$A \equiv A_\mu dx^\mu = h(r)(d\tau - 2n \cos \theta d\phi) , \quad (26)$$

where:

$$h(r) = \frac{qr}{r^2 - n^2} + v \frac{r^2 + n^2}{r^2 - n^2} . \quad (27)$$

Conditions of smoothness of the Euclidean section will result in the parameter  $q$  being related to the parameter  $v$  (analogous to  $m$  being related to  $n$  in the familiar neutral case), giving us a deformation from the uncharged system. Setting them to zero recovers the solutions we discussed above.

The analogous conditions for the smoothness of the Euclidean section were discussed in refs. [54, 55], and some thermodynamic quantities discussed there and in ref. [56], including (for the latter case) a computation of the action for the charged Taub–NUT and Bolt cases. We will not review all this here, since the ideas are very similar to those we presented in an earlier section, with the addition of the condition that the gauge potential  $A$  must be regular at the position of the nut or bolt at  $r = r_+$  where  $F(r_+) = 0$ . We simply list below some of the results that we need, in order to discuss the phase structure (which has not been explored in the literature). The mass can be written as:

$$m = \frac{r_+^2 + n^2 + 4n^2v^2 - q^2}{2r_+} + \frac{1}{2\ell^2} \left( r_+^3 - 6n^2r_+ - 3\frac{n^4}{r_+} \right) , \quad (28)$$

with

$$q = -v \frac{r_+^2 + n^2}{r_+} . \quad (29)$$

The electric charge and potential at infinity given by

$$Q = q , \quad \Phi = -\frac{qr_+}{r_+^2 + n^2} = v . \quad (30)$$

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<sup>13</sup>In fact, it is natural to study dilatonic versions as well, for example for non-trivial embedding into string theory, as first done using exact heterotic string conformal field theory techniques in ref. [49], and using T- and S-duality transformations in ref. [50]. The result is that there an axion charge turns on as well, and so the system becomes triply dyonic, with the dilaton and axion being the electric/magnetic pairs. We will not study the addition of such fields in this paper.

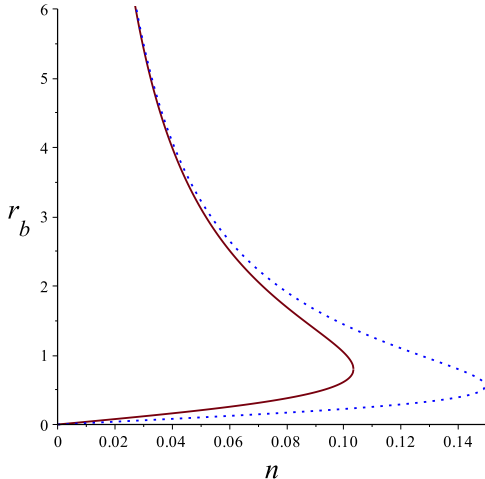


Figure 5: The available dyonic bolt radii, the union of an upper and a lower branch that connect at  $n_{\max}^{(v)} \simeq 0.103\ell$ . Here we chose  $\ell = 1$  and  $v = 1$ . The dotted line is the neutral case for comparison. See text for discussion.

line of critical points in the phase diagram ended in a new second order critical point. One might wonder if that can happen here. Indeed, there are two branches of new solutions that appear, but they do not add (in the Euclidean section) any new physical structures: Taking the small  $r_b/\ell$  limit of equation (31) gives the cubic  $r_b^3 - 2n(1+v^2)r_b^2 + 2v^2n^3 = 0$ . At  $v = 0$  the two new branches are the repeated (unphysical) root  $r_b = 0$ , joining our familiar large  $\ell$  solution  $r_b = 2n$ . Turning on  $v$  then shows that the two branches grow out as a pair of roots from the origin of the  $(r_b, n)$  plane, one positive, the other negative. It can be seen that both have  $r_b < n$ , and so are not physical.

Instead we have a system of physical solutions that is similar in structure to what we saw before (see figure 5 for an example; *c.f.* figure 1). There are two branches of bolts, a “small” and a “large”, and they join at a deformed value of a maximum  $n$  we can call  $n_{\max}^{(v)}$ , which is less than  $n_{\max}$ . Correspondingly, we might expect that the transition from the dyonic Taub–NUT to the dyonic Taub–Bolt is at lower  $n$ , or higher temperature, and we will confirm this below.

Since there are no new branches of solution for  $r_b$  in the presence of charge, it seems that it qualitatively does not matter too much whether we use the fixed charge or fixed potential ensemble to study the phase structure although of course there are differences in the precise values of the parameters at transition. We can sketch out what happens, picking the fixed potential ensemble<sup>14</sup>.

The action of the system was computed in ref. [56] using an extension of the techniques

<sup>14</sup>This is the easier ensemble to study in comparing the NUT and Bolt cases since fixing  $v$  is easier than fixing  $q$  when comparing the two solutions, because (looking at equation (29))  $q$  depends upon  $r_b$ , which is a complicated function of  $v$ .

The solution is a dyonic nut if  $r_+ = r_n = n$ , and a dyonic bolt if  $r_+ = r_b$  satisfies the fourth order equation:

$$6nr_b^4 - \ell^2 r_b^3 + 2n(\ell^2 - 3n^2 + \ell^2 v^2)r_b^2 - 2\ell^2 v^2 n^3 = 0 . \quad (31)$$

This is a more complicated condition than our quadratic in equation (6). One might wonder if there is potential for certain kinds of new structure here, in analogy with what happens for charged black holes. In that case, the more complicated equation for the horizon radius in the presence of charge produced a new physical branch of solutions at small horizon radius and high temperature that gave a new phase structure in the fixed charge ensemble [27, 28]. The

outlined in refs. [38, 57], with the result (in the fixed potential ensemble):

$$I = -\frac{2\pi}{G} \frac{[r_+^4 - \ell^2 r_+^2 + n^2(3n^2 - \ell^2)]r_+^2 - (r_+^4 + 4n^2 r_+^2 - n^4)\ell^2 v^2}{(3r_+^2 - 3n^2 + \ell^2)r_+^2 + (r_+^2 - n^2)\ell^2 v^2} . \quad (32)$$

The entropy was computed as:

$$S = \frac{2\pi}{G} \frac{[3r_+^4 + (\ell^2 - 12n^2)r_+^2 + n^2(\ell^2 - 3n^2)]r_+^2}{(3r_+^2 - 3n^2 + \ell^2)r_+^2 + (r_+^2 - n^2)\ell^2 v^2} + \frac{2\pi}{G} \frac{(r_+^4 + 4n^2 r_+^2 - n^4)\ell^2 v^2}{(3r_+^2 - 3n^2 + \ell^2)r_+^2 + (r_+^2 - n^2)\ell^2 v^2} . \quad (33)$$

Let us pause to see how this all looks in the case of the dyonic nut, where the expressions are much simpler. We can put  $r_+ = n$  to get:

$$m_n = n - \frac{4n^3}{\ell^2} , \quad Q = -\frac{2nv}{G} , \quad \Phi = v , \quad (34)$$

and

$$I_n = \frac{4\pi n^2}{G} \left(1 - \frac{2n^2}{\ell^2} + 2v^2\right) , \quad S_n = \frac{4\pi n^2}{G} \left(1 - \frac{6n^2}{\ell^2} + 2v^2\right) . \quad (35)$$

Interestingly, the mass is the same as in the neutral case. This means that the enthalpy is unchanged in the extended thermodynamics, which leads to another new result: A quick computation shows that the thermodynamic volume  $V_n$  for the dyonic Taub–NUT stays the same as it was for the neutral case. Put differently, the shift in the entropy is entirely accounted for by the new  $-Q\Phi/2$  term in the Smarr [58] relation:

$$\frac{H}{2} - TS + pV - \frac{Q\Phi}{2} = 0 . \quad (36)$$

Just as in ref. [25] for the neutral case, we can use the same Smarr relation to deduce the thermodynamic volume for the dyonic bolt case. The resulting expression does depend (explicitly and implicitly through  $r_b$ 's dependence on  $v$ ) on the deformation parameter  $v$  and it is long and not useful for us here, so we will not write it.

## 6 Deformed Phase Structure

From our Taub–NUT–dyon's entropy, we see that the specific heat at constant pressure is now:

$$C_p^{(v)} = -\frac{8\pi n^2}{G} \left(1 - \frac{12n^2}{\ell^2} + 2v^2\right) . \quad (37)$$

So the deformation parameter  $v$  has shifted the location of the place in the  $(p, T)$  plane where the entropy is negative, and where the specific heat goes negative. Our previously obtained temperatures for the borders of the positive wedge get rescaled by a factor  $(1 + 2v^2)^{-1/2}$ . The temperature where the entropy starts going positive is lower, as well the temperature where the specific heat  $C_p$  becomes negative. The dyonic charge evidently hastens the destabilisation of Taub–NUT. In the limit of infinite  $v$ , entropy becomes positive for all temperatures, but then  $C_p < 0$  everywhere.

The phase structure for the dyonic system can now be explored, although analytic expressions do not seem to be possible for all regions. A numerical exploration reveals some pleasant results, however. For example, figure 6 shows a plot of the action difference between dyonic Taub–Bolt and dyonic Taub–NUT as a solid curve, against the neutral case for comparison (a dashed curve). They have the same value of the pressure ( $\ell = 1$  and we’ve chosen units such that  $G = 1$ ) and  $v = 1$  for the dyon. As anticipated, the transition temperature has been raised.

A series of such examples can be generated numerically for varying pressure, extracting the transition temperature for each case. This allows for the generation of a new phase diagram for the dyonic deformation as a companion to the neutral one we generated in figure 3. The result is shown in figure 7(a). There seems to be no simple analytic expression for the shape of the first order transition line for non-zero  $v$ , although it fits the  $T^2$  dependence rather well.

The deformation widens the size of the unstable  $C_p < 0$  region. The fate of the phase diagram in the large  $v$  limit is therefore interesting, since in that case the  $C_p < 0$  region increasingly envelops the phase diagram, at the expense of the negative entropy region, which shrinks away, along with the positive wedge, against the  $p$ -axis.

Although its full meaning is unclear, it is worth noting that the case where we continue  $v$  to being purely imaginary is also interesting<sup>15</sup>. From equations (35) and (37), we see that the positive wedge is now pushed more toward the right, toward the transition line. A numerical exploration shows that the temperature at which there are available bolt solutions is decreased (*i.e.*  $n_{\max}^{(v)}$  is now greater than  $n_{\max}$ ). Correspondingly, the first order phase transition line tilts more toward the left, and so approaches the positive wedge for increasing imaginary  $v$ . The  $C_p < 0$  region for Taub–NUT gets squeezed. See figure 7(b).

Ultimately there are yet new regimes that appear (for increasingly large imaginary  $v$ ). One occurs when the  $C_p < 0$  region no longer has lowest action. In other words, the transition line

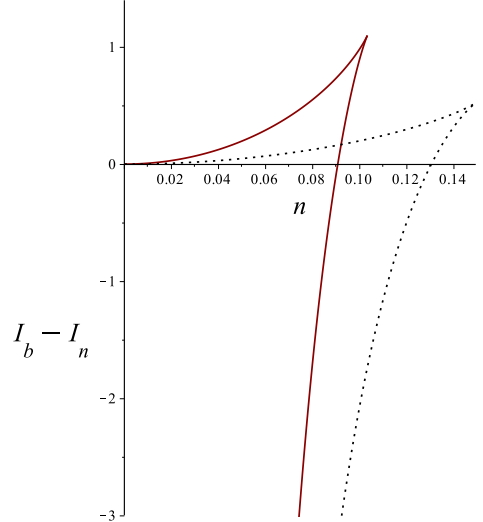


Figure 6: The solid curve is the difference between the action for dyonic Taub–Bolt and dyonic Taub–NUT in the fixed potential ensemble. The dotted curve is the neutral case, for comparison. The transition temperature has increased. Here we used  $\Phi = v = 1$ , and  $\ell = 1$ .

<sup>15</sup>Here, we are perhaps in good company: The Lee–Yang edge singularity, an important model in the study of phase transitions, arises from continuing the Ising model in an external magnetic field to regimes where the magnetic field is imaginary [59,60]. Imaginary  $v$  here corresponds to imaginary magnetic and electric fields for our dyons.



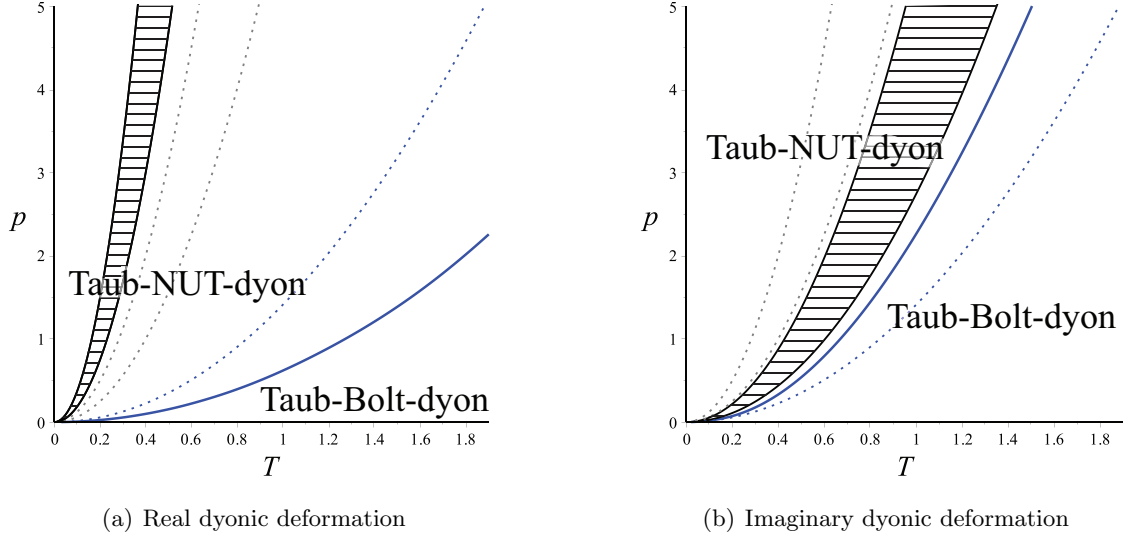


Figure 7: The phase diagram for the Taub–NUT–dyon and Taub–Bolt–dyon phases, for (a) real dyon deformation parameter  $v^2 = 1$ , and (b) complex dyon parameter  $v^2 = -9/32$ . The “positive wedge” (shaded region) has shifted left and the (solid) transition line has tilted right. See text for explanation. The dotted lines are for comparison to the neutral case in figure 3.

to Taub–Bolt is inside the positive wedge:  $T^* < T^o$ . In this case the transition line is genuinely a coexistence line analogous to that found in solid/liquid systems. Another regime is when (at  $v^2 = -1/2$ ) the  $C_p < 0$  Taub–NUT region vanishes entirely. The whole positive wedge disappears, leaving a phase diagram where there is only Taub–NUT with negative entropy before the transition line to Taub–Bolt. Interestingly, yet another new phase can appear at even higher imaginary  $v$ , as illustrated by the samples in figures 8(a) and 8(b).

The large Taub–Bolt solutions themselves begin to develop regions of negative entropy (as is evident from the form of equation (33)). The sample plots of the action difference show that for large enough  $|v|$  the large Taub–Bolt solutions become thermodynamically favoured as soon as they become available, as their action relative to the Taub–NUT solutions are negative for all  $n < n_{\max}^{(v)}$ . So these negative entropy Taub–Bolts appear on the phase diagram for a window of temperatures, until at higher temperatures the positive entropy Taub–Bolt solutions take over again. Perhaps there is even more interesting and instructive physics to explore further in this regime.

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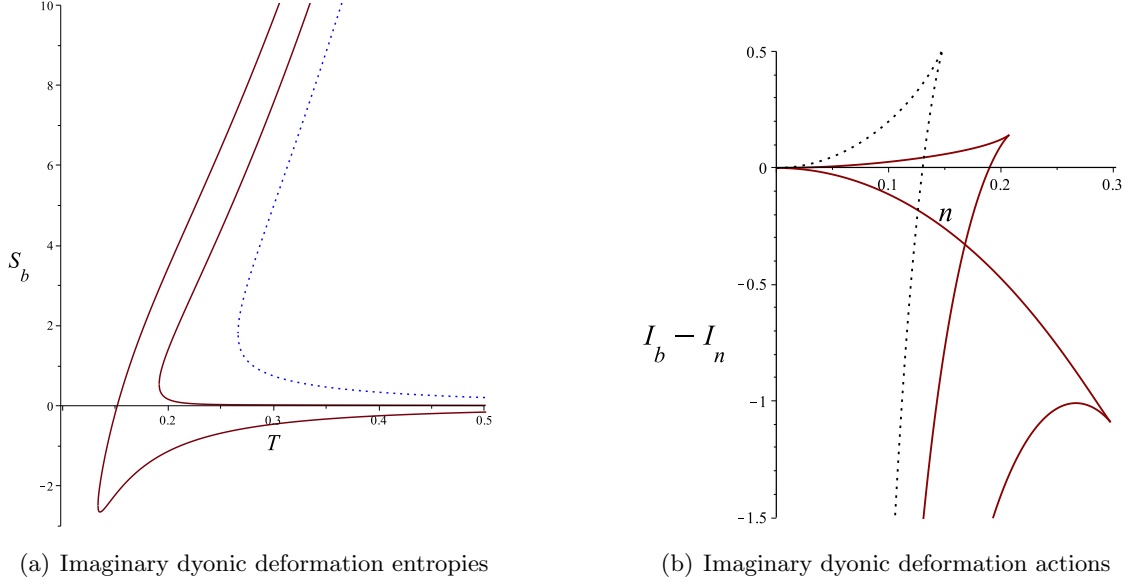


Figure 8: Some examples of curves showing the dyonic Taub-Bolt entropy (*vs.*  $T$ ) and action difference (*vs.*  $n$ ). The dotted lines are for comparison to the neutral case. We have  $G = \ell = 1$  in all plots. In plots (a), the curves, reading from left to right, have  $v^2 = -1, -1/2, 0$ , which are the values of the curves reading right to left in plots (b).

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